Conformal invariance, multifractality, and finite-size scaling at Anderson localization transitions in two dimensions

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(Received 2 December 2009; revised manuscript received 9 April 2010; published 19 July 2010)

We generalize universal relations between the multifractal exponent \(a_0\) for the scaling of the typical wave-function magnitude at a (Anderson) localization-delocalization transition in two dimensions and the corresponding critical finite-size-scaling (FSS) amplitude \(\Lambda_c\) of the typical localization length in quasi-one-dimensional (Q1D) geometry: (i) when open boundary conditions are imposed in the transverse direction of Q1D samples (strip geometry), we show that the corresponding critical FSS amplitude \(\Lambda^c_0\) is universally related to the boundary multifractal exponent \(a^c_0\) for the typical wave-function amplitude along a straight boundary (surface). (ii) We further propose a generalization of these universal relations to those symmetry classes whose density of states vanishes at the transition. (iii) We verify our generalized relations [Eqs. (6) and (7)] numerically for the following four types of two-dimensional Anderson transitions: (a) the metal-to-(ordinary insulator) transition in the spin-orbit (symplectic) symmetry class, (b) the metal-to-\(\mathbb{Z}_2\) topological insulator transition which is also in the spin-orbit (symplectic) class, (c) the integer quantum-Hall plateau transition, and (d) the spin quantum-Hall plateau transition.

DOI: 10.1103/PhysRevB.82.035309

PACS numbers: 73.20.Fz, 05.45.Df, 72.15.Rn

I. INTRODUCTION

Localization-delocalization (LD) or Anderson localization transitions of noninteracting electrons are continuous phase transitions driven by disorder.1–5 When disorder is weak, the single-electron wave functions are extended over the whole sample. Sufficiently strong disorder localizes electrons within a finite region in space. The linear size of this region is the localization length \(\xi\) characterizing the typical size of the wave functions \(\psi(r)\).1 As the disorder strength is reduced, the localization length increases and eventually diverges at an LD transition point. The localization length is the analogue of the correlation length at nonrandom continuous phase transitions. At the LD transition point, wave-function amplitudes obey scale-invariant, multifractal statistics,6–9 that is, the disorder-averaged \(q\)th moment of the square of absolute value of wave function has a power-law dependence on the linear dimension \(L\) of the system, with an exponent that is a nonlinear function of \(q\).

Let us recall that continuous phase transitions in nonrandom systems are known to be quite generally described by conformally invariant field theories. Conformal symmetry is especially powerful in two dimensions (2D), where its presence leads to an infinite number of symmetry constraints. This, in many cases, allows for a rather complete description of critical properties.10,11 Effective (field) theories describing the random LD transitions are also expected to possess conformal symmetry. In fact, we have recently shown by numerical simulations of a standard LD transition occurring in two dimensions, namely, of the metal-insulator transition in the 2D spin-orbit (symplectic) symmetry class,12 that multifractal exponents of critical wave functions evaluated on a straight boundary and those at a corner are related through a simple relation dictated by conformal symmetry.13

Conformal symmetry is known to impose strong constraints on finite-size scaling (FSS) for phase transitions in nonrandom systems with quasi-one-dimensional (Q1D) geometry. For these systems Cardy has shown14 that the correlation length \(\xi\) which characterizes the decay of the two-point correlation function of any (conformal primary) operator along a cylinder or a strip of width \(M\), is related to the bulk \(x_b\) or surface \(x_s\) scaling dimension of the operator in two dimensions through

\[
\frac{M}{\xi} = \begin{cases} 
2\pi x_b, & \text{cylinder(periodic BC)}, \\
\pi x_s, & \text{strip(open BC)}. 
\end{cases}
\]

Here BC stands for boundary conditions imposed in the transverse direction.

The generalization of Eq. (1) to scale-invariant disordered 2D systems was first provided in the study of random 2D diluted ferromagnets in Ref. 16 [for the 2D bulk exponents and Q1D cylinder geometry (periodic BCs)]. In a random system the scaling of an observable (such as, for example, a “spin”) is in general characterized by the set of scaling dimensions \(x_q\) of all its \(q\)th moment disorder averages. Equation (1) generalizes16 to all these moments. In particular, the correlation length \(\xi_q\) characterizing the exponential decay of the \(q\)th moment of a correlation function of the observable in Q1D cylinder geometry is related to the 2D scaling exponent by

\[
\frac{M}{\xi_q} = 2\pi x_q, \quad \text{cylinder(periodic BC)}. \tag{2}
\]

At the same time, by using an expansion about \(q=0\) of the \(q\)th moments in the 2D system and in the Q1D cylinder geometry, it was demonstrated in Ref. 16 that such a relationship holds also for the corresponding “typical” quantities.
referring to a fixed disorder realization. In particular, if $a_0$ denotes the typical 2D bulk scaling dimension of the observable, and if $1/\xi$ denotes the Lyapunov exponent characterizing the inverse of the typical Q1D correlation length in cylinder geometry, then again the relationship
\[ \frac{M}{\xi} = 2\pi a_0, \quad \text{cylinder(periodic BC)} \] (3)
holds.

Later, Refs. 9 and 17 proposed a corresponding formula in the context of LD transitions in two dimensions,
\[ \frac{M}{\xi_p} = 2\pi(a_0^b - 2) \] (4)
[the shift by two between the right-hand side of Eqs. (4) and (3) arises from different conventions]. Here, $\xi_p$ is the typical Q1D localization length in cylinder geometry (the subscript $p$ of $\xi_p$ denotes periodic BCs imposed in the transverse direction). The exponent $a_0^b$ in Eq. (4) characterizes the scaling of a typical critical wave-function amplitude in the bulk of a 2D system of linear dimension $R$,
\[ \overline{\ln|\psi(r)|^2} = -a_0^b \ln R, \] (5)
where the overbar stands for the disorder average. Equation (4) has been confirmed numerically for the integer quantum-Hall (IQH) plateau transition 9,17,18 and for the 2D metal-insulator transition in the spin-orbit (symplectic) symmetry class 13,19,20.

We note that the relation (4), in the form presented, is only valid for systems in which the average bulk density of states (DOS) is constant and nonvanishing at the transition. This is the case for LD transitions in the three Wigner-Dyson classes. These include the IQH plateau transition and the LD transition in the spin-orbit (symplectic) class. However, as is now well known, there are symmetry classes in which the DOS vanishes at the transition. This is the case, for example, for the so-called spin quantum-Hall transition of the Bogoliubov-de Gennes (BdG) quasiparticles in symmetry class C21–23 (in the nomenclature of Ref. 25).

In this paper we derive a generalization of the relationship (4) between the exponent $a_0^b$ and the typical Q1D correlation length $\xi_p$ for LD transitions in 2D with a vanishing critical DOS. The result is
\[ \frac{M}{\xi_p} = 2\pi(a_0^b - 2 + x_p), \] (6)
where the exponent $x_p$ characterizes the critical behavior of the (bulk) DOS ($x_p = 0$ in the Wigner-Dyson classes).

Furthermore, we derive a FSS formula for the typical Q1D localization length, when open BCs are imposed in the transverse direction. The specific open BC we consider in this paper is a reflecting BC which means that the system simply ends at the boundary so that there is no current flowing across the boundary. The second line of Eq. (1) suggests that the localization length should be related to a surface exponent characterizing multifractality of critical wave functions near boundaries of disordered systems.24 Indeed, our result is the formula
\[ \frac{M}{\xi} = \pi(a_0^b - 2 + x_p), \] (7)
where now $a_0'$ is the surface (i.e., boundary) exponent characterizing scaling of a typical wave-function amplitude near a straight (reflecting) boundary. $a_0'$ is defined in the same way as $a_0^b$ in Eq. (5), except that now the point $r$ is close to a straight boundary of the 2D system of linear dimension $R$.

The typical Q1D localization length $\xi$, is computed in the geometry of a strip of width $M$ with open (reflecting) BCs imposed in the transverse direction (the subscript $o$ stands for “open”).

The organization of this paper is as follows. In Sec. II we derive Eqs. (6) and (7). In Sec. III we verify both these equations numerically by computing the critical FSS amplitude ($\Lambda_p$ or $\Lambda_o$) of the typical Q1D localization length, defined as
\[ \Lambda_p = \frac{2\xi_p}{M}, \quad \Lambda_o = \frac{2\xi_o}{M} \] (8)
for both types of BCs (the factor 2 in this definition is standard convention). We verify Eq. (7) for (a) the metal-to-(ordinary) insulator transition in the spin-orbit (symplectic) class [class AII of Ref. 25], (b) the LD transition between a metal and a $Z_2$ topological insulator in the “quantum spin-Hall” (QSH) effect25 which also belongs to the spin-orbit (symplectic) class [class AII of Ref. 25], [c] the IQH plateau transition in the unitary symmetry class [class A of Ref. 25]. The bulk relation, Eq. (6), was already verified for systems (a)–(c), where $x_p = 0$, in previous work 9,13,17–20. We finally verify numerically Eqs. (6) and (7) for the spin quantum-Hall transition in symmetry class C of Ref. 25. Table I summarizes the numerical results presented in detail in Sec. III. Section IV presents our conclusions.

II. LOCALIZATION LENGTH AND MULTIFRACTALITY

In this section we provide a derivation of Eqs. (6) and (7). Let us begin with a brief discussion of the underlying assumptions. We are interested in scaling properties of the disorder average of some physical observable [e.g., the local DOS (LDOS)] at an LD transition point. One can recast this disorder average into a statistical average of a properly defined operator $O$ in a certain field theory (e.g., a replica or supersymmetric nonlinear sigma model).2,4 The scaling properties of $O$ at the critical point are then controlled by the fixed point of the renormalization-group (RG) flow of the corresponding field theory. We are now ready to state the two important assumptions we make in our derivation:27 (1) the fixed-point theory is a conformal field theory. (2) At the fixed point of the RG transformation, the operator $O$ is a primary field operator in the conformal field theory.

A. Finite-size scaling in cylinder geometry and bulk exponents

Let us consider a disordered electronic system at its critical point, confined to a disk of radius $R$ in the 2D $x$-$y$ plane, or equivalently, the complex plane with the coordinate $z = x + iy$. We assume that all along the boundary of the disk there
is a metallic electrode attached, thus allowing for the electron in the system to escape.28 This (absorbing) boundary condition introduces a finite broadening η of the single-particle levels in the system. We assume that the broadening is on the order of the mean level spacing in the system. This provides a regularization for Green’s function and the LDOS as follows:

\[ G_z(z, z'; E) = \sum_n \frac{\psi_n^*(z) \psi_n(z')}{E - E_n + i\eta} \tag{9} \]

\[ \rho_E(z) = \frac{i}{2\pi} \left[ G_i(z, z; E) - G_c(z, z; E) \right] \]

\[ = \frac{1}{\pi} \sum_n \frac{|\psi_n(z)|^2 \eta}{(E - E_n)^2 + \eta^2}. \tag{10} \]

Here the wave functions \( \psi_n(z) \) of the closed system are normalized in the disk: \( \int_{|z|\leq R} |\psi(z)|^2 d^2z = 1 \). The integral of the LDOS \( \rho_E(z) \) over the disk gives the global DOS \( \rho_E \) multiplied by the disk area \( \pi R^2 \).

Statistical properties of metallic or critical wave functions at energy \( E \) are closely related to those of the LDOS.8 In particular, if we are interested in the scaling of the moments of such wave functions and the moments of the LDOS, we can write symbolically

\[ |\psi_n(z)|^2 \approx \frac{\rho_E(z)}{\pi R^2 \rho_E}. \tag{11} \]

Disorder averages of powers of the LDOS \( \rho_E(z) \) (as well as those of products of Green’s function) are represented by expectation values of operators in the corresponding field theory.3,4 We denote this by

\[ \left[ \rho_E(z) \right]^g \sim \langle O_q(z) \rangle, \tag{12} \]

where the angular brackets denote the expectation value in the field theory. Here \( O_q(z) \) is the operator which corresponds to the \( q \)th moment of \( \rho_E(z) \). (We point out that here and in what follows the power \( q \) can take any real values.29)

In view of Eq. (11), the same operator represents moments of the wave function \( \psi_n(z) \),

\[ (R^2 \rho_E)^{\frac{1}{2} g} |\psi_n(z)|^{2g} \sim \langle O_q(z) \rangle. \tag{13} \]

Notice that the global DOS is self-averaging and can be pulled out of the disorder average along with powers of the radius \( R \). The product \( R^2 \rho_E \propto \delta^1 \), where \( \delta \) is the mean level spacing in the disk.

Now we concentrate on the wave functions and the DOS at the critical energy, \( E = E_c \), and drop the subscript \( E \). The global DOS \( \rho \) may vanish at criticality in the infinite system. In a finite system the disorder-averaged \( \rho \) always has a power-law behavior,

\[ \rho \sim R^{-\nu_c}, \tag{14} \]

where the exponent \( \nu_c \) vanishes in the standard Wigner-Dyson classes but is known to be nonzero in other symmetry classes. For example, at the (2D) spin quantum-Hall transition in symmetry class C,21–23 the exact value is known, \( \nu_c = 1/4 \).21

We now make use of the previously stated assumptions27 of conformal invariance and the fact that \( O_q \) is a primary conformal scaling operator with the bulk scaling dimension \( x^b_q \) at the LD transition. If we choose a point \( |z| \ll R \) close to the origin of the disk, then the one-point function (the field theory expectation value) scales as

\[ \langle O_q(z) \rangle \sim R^{-x^b_q}. \tag{15} \]

Combining this with Eqs. (13) and (14), we obtain the scaling of the moments of the critical wave functions,

\[ |\psi(z)|^{2g} \sim R^{2g-x^b_q q x_p}. \tag{16} \]

for \( |z| \ll R \). Notice that the exponent of \( R \) on the right-hand side should vanish at \( q = 0 \), and should be \(-2\) at \( q = 1 \) due to the normalization of the wave function. These conditions determine

\[ x^b_0 = 0, \quad x^b_1 = x_p. \tag{17} \]

Some important details of the definition and properties of multifractal exponents are in order here. A slightly more detailed (coarse-grained) description of multifractal wave functions (in 2D) involves breaking the system into little square boxes \( B_i \) of size \( r \times r \) labeled by \( i \). The number of these boxes \( N \) scales as \( N \sim (R/r)^2 \). One then calculates the probability \( p_i \) for an electron to be in the \( i \)th box as

\[ p_i = \int_{B_i} |\psi(z)|^2 d^2z \tag{18} \]

and forms the so-called average generalized inverse participation ratios,
where the set of exponents $\tau_q$ is usually referred to as the multifractal spectrum.

Note that the probabilities $p_i$ whose moments enter the definition of $P_q$ are bounded by $0 \leq p_i \leq 1$. This bound implies that $P_q$ must be a nonincreasing function of $q$ since $p_i^{q_1} \leq p_i^{q_2}$ for $q_1 < q_2$. Moreover since $p^q = \exp(q \ln p)$ is convex as a function of $q$, the same is true for $P_q$. Then the multifractal spectrum $\tau_q$ in Eq. (20) must be a nondecreasing, concave function of $q$. Generally speaking, there may be a value of $q=q_f$ where $\tau_q$ has a horizontal tangent. Then it follows that $\tau_q = \text{const}$ for $q \geq q_f$. Such change in the behavior of $\tau_q$ from an increasing function to a constant is often referred to as “freezing” or “termination” (see Ref. 5 for more details). In all known cases the value $q_f$ where such termination occurs satisfies $q_f > 0$. Then we can safely use Eq. (16), and similar equations in the following sections, in the vicinity of $q=0$ without worrying about a possible termination transition.

Expanding both sides of Eq. (16) in $q$ about $q=0$ yields the typical scaling exponent, Eq. (5), where

$$a_0^b = 2 - x_p + \frac{d x_b^b}{d q} \Bigg|_{q=0}.$$  \hspace{1cm} (21)

Next, let us consider the conformal mapping

$$w = \frac{M}{2 \pi} \ln z, \quad z = \exp \left( \frac{2 \pi}{M} w \right),$$  \hspace{1cm} (22)

which maps the disk to the semi-infinite cylinder of circumference $M$ in the complex $w$ plane,

$$w = u + i v, \quad u \leq L = \frac{M}{2 \pi} \ln R, \quad 0 \leq v < M$$  \hspace{1cm} (23)

with an absorbing boundary condition at $u=L$. The assumption that $O_q$ is a primary conformal operator allows us to relate its expectation value on the cylinder to that in the disk,

$$\langle O_q(w) \rangle = \left| \frac{d z}{d w} \right|^{x_b^s} \langle O_q(z) \rangle \sim \left( \frac{2 \pi}{M} \right)^{x_b^s} \exp \left[ - \frac{2 \pi}{M} x_b^s (L - u) \right].$$  \hspace{1cm} (24)

This immediately gives the moments of the LDOS in the cylinder

$$\langle \rho(w) \rangle q \sim \exp \left[ - \frac{2 \pi}{M} x_b^s (L - u) \right].$$  \hspace{1cm} (25)

From the exponential decay of the moment $\langle \rho(w) \rangle q$ away from the end of the semi-infinite cylinder in Eq. (25), for sufficiently small positive values of $q$, we identify the "$q$-dependent localization length" $\xi_q(q)$ in the cylinder geometry as

$$\xi_q(q) = \frac{M}{2 \pi x_b^s}.$$  \hspace{1cm} (26)

(Here "$p" denotes again the "periodic" BCs of the cylinder.) The typical Q1D localization length $\xi_q$ in cylinder geometry is read off from the typical exponential decay of the LDOS away from the end of the semi-infinite cylinder,

$$\ln \rho(w) = - \frac{L - u}{\xi_q} + \cdots.$$  \hspace{1cm} (27)

Expanding again Eq. (25) in $q$ about $q=0$ yields

$$\frac{M}{\xi_q} = 2 \pi \frac{d x_b^s}{d q} \bigg|_{q=0} = 2 \pi (a_0^b - 2 + x_p),$$  \hspace{1cm} (28)

where we have used Eq. (21). This is our previously mentioned result, Eq. (6), which generalizes Eq. (4) to all symmetry classes, including those with critical DOS.

In Sec. III D we numerically verify Eq. (6) for the spin quantum-Hall effect (symmetry class C) by computing numerically the FSS amplitude $\Lambda_p = 2 \xi_p^b / M$ of the typical Q1D localization length $\xi_p$ in cylinder geometry; according to our above-obtained result (28) this quantity is predicted to equal

$$\Lambda_p = \frac{1}{\pi (a_0^b - 2 + x_p)}$$  \hspace{1cm} (29)

with $x_p = 1/4$.

**B. Finite-size scaling in strip geometry and surface (boundary) multifractal exponents**

We now apply the same arguments to discuss finite-size scaling in the presence of open (reflecting) BCs in the transverse direction (strip geometry).

For this purpose we first consider the operator $O_q$ placed close to the origin in the interior of the half-disk $|z| \leq R$, $\text{Im} \ z \geq 0$. The boundary of the system on the real axis is assumed reflecting, and the rest is attached to a metallic lead, as in the previous section. In this situation the expectation value of $O_q(z)$ for $|z| \ll R$ is given by

$$\langle O_q(z) \rangle \sim R^{-x_b^s},$$  \hspace{1cm} (30)

where the boundary scaling dimension $x_b^s$ (the superscript $s$ stands for "surface") is typically different from the bulk dimension $x_b^b$. In analogy with Eq. (16) we now have, upon making again use of Eq. (13),

$$\langle \rho(z) \rangle q \sim R^{-2q-x_b^s+q x_p},$$  \hspace{1cm} (31)

where the same exponent $x_p$ (a bulk exponent) enters through the global DOS. Note that Eq. (31) still implies $x_b^s = 0$ but now, in the boundary case, there is no restriction on $x_b^s$ (in contrast to the bulk case: see Eq. (16) and the subsequent text). Also, in complete analogy to the bulk case, the exponent of $R$ in Eq. (31) must be a monotonic function of $q$. Upon expanding both sides of Eq. (31) in $q$ about $q=0$, one
TABLE II. A list of parameters obtained or used in the FSS analysis for the scaling functions defined in Eqs. (41) and (57). Here $\Lambda_\nu$, $\nu$, and $y$ are obtained through fitting. $N_d$ and $N_p$ denote the numbers of data points and fitting parameters used in the fitting procedure, respectively. The fitting functions are truncated at the orders $P$ and $Q$. $\chi^2$ and $g$ denote the values of chi squared and the goodness of fit probability, respectively.

<table>
<thead>
<tr>
<th>System</th>
<th>BCs</th>
<th>Scaling function</th>
<th>$\Lambda_\nu$</th>
<th>$\nu$</th>
<th>$y$</th>
<th>$N_d$</th>
<th>$N_p$</th>
<th>$P$</th>
<th>$Q$</th>
<th>$\chi^2$</th>
<th>$g$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Symplectic (M-I)</td>
<td>Reflecting</td>
<td>Eq. (41)</td>
<td>$1.50 \pm 0.01$</td>
<td>$2.79 \pm 0.03$</td>
<td>$-1.03 \pm 0.03$</td>
<td>$85$</td>
<td>$9$</td>
<td>$2$</td>
<td>$2$</td>
<td>$86.2$</td>
<td>$0.2$</td>
</tr>
<tr>
<td>Symplectic (M-QSH)</td>
<td>Reflecting</td>
<td>Eq. (41)</td>
<td>$7.20 \pm 0.01$</td>
<td>$-0.81 \pm 0.08$</td>
<td>$8$</td>
<td>$3$</td>
<td>$0$</td>
<td>$0$</td>
<td>$7.2$</td>
<td>$0.2$</td>
<td></td>
</tr>
<tr>
<td>IQH</td>
<td>Reflecting</td>
<td>Eq. (41)</td>
<td>$1.624 \pm 0.002$</td>
<td>$2.55 \pm 0.01$</td>
<td>$-1.29 \pm 0.04$</td>
<td>$134$</td>
<td>$6$</td>
<td>$3$</td>
<td>$2$</td>
<td>$144.0$</td>
<td>$0.2$</td>
</tr>
<tr>
<td>SQH in class C</td>
<td>Periodic</td>
<td>Eq. (57)</td>
<td>$0.8189 \pm 0.0004$</td>
<td>$1.335 \pm 0.016$</td>
<td>$-0.94 \pm 0.01$</td>
<td>$73$</td>
<td>$8$</td>
<td>$2$</td>
<td>$2$</td>
<td>$56.2$</td>
<td>$0.7$</td>
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<tr>
<td>SQH in class C</td>
<td>Reflecting</td>
<td>Eq. (41)</td>
<td>$1.101 \pm 0.002$</td>
<td>$1.335 \pm 0.005$</td>
<td>$-1.05 \pm 0.02$</td>
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<td>$9$</td>
<td>$3$</td>
<td>$2$</td>
<td>$86.1$</td>
<td>$0.4$</td>
</tr>
</tbody>
</table>

obtains the scaling exponent $\alpha_0$ of the typical wave-function amplitude at the boundary,

$$\ln|\phi(z)|^2 \sim -\alpha_0 \ln R,$$

where now

$$\alpha_0 = 2 - x_p + \frac{dx_q}{dq} \bigg|_{q=0}. \quad (33)$$

Next, in order to relate this to the strip geometry, we use the conformal transformation

$$w = \frac{M}{\pi} \ln z, \quad z = \exp\left(\frac{\pi}{M} w\right), \quad (34)$$

which maps the half disk to a semi-infinite strip of width $M$ in the $w$ plane

$$w = u + iv, \quad u \leq L = \frac{M}{\pi} \ln R, \quad 0 \leq v \leq M. \quad (35)$$

The expectation value on the strip now follows again since $O_q$ as a primary conformal operator, transforms simply under conformal transformations,

$$\langle O_q(w) \rangle = \left[ \frac{dz}{dw} \right]^{\nu_q} \langle O_q(z) \rangle \sim \left(\frac{\pi}{M}\right)^\nu_q \exp\left[-\frac{\pi}{M} v_q(L-u)\right]. \quad (36)$$

From this we obtain the exponential decay of the moments of the LDOS away from one end of the strip,

$$\bar{\rho}(w) \sim \exp\left[-\frac{\pi}{M} v_q(L-u)\right]. \quad (37)$$

As in the bulk case, the exponential decay of the right-hand side in Eq. (37), for sufficiently small positive values of $q$, gives the “$q$-dependent Q1D localization length” along the strip

$$\xi_0(q) = \frac{M}{\pi v_q}. \quad (38)$$

As before, the typical Q1D localization length $\xi_0$ in strip geometry is obtained by expanding both sides of Eq. (37) in $q$ about $q=0$,

$$M \frac{\xi_0}{\xi_0} = \frac{dx_q}{dq} \bigg|_{q=0} = \pi(\alpha_0 - 2 + x_p), \quad (39)$$

where we have used Eq. (33). This is our previously announced result from Eq. (7).

In subsequent sections we verify Eq. (7) for various LD transitions by computing numerically the FSS amplitude $\Lambda_alpha = 2\xi_0/M$ of the Q1D typical correlation length $\xi_0$ on the strip ($\sigma$-open, reflecting BCs) which, according to our result, is predicted to be equal to

$$\Lambda_\nu = \frac{2}{\pi(\alpha_0 - 2 + x_p)}. \quad (40)$$

III. NUMERICAL RESULTS

In this section we present the results of our numerical simulations supporting Eqs. (29) and (40). For convenience, we have gathered all the relevant fitting parameters and other numerical data in a single Table II.

In this section we have to distinguish off-critical and critical values of the Q1D localization lengths, $\xi$ and $\xi_0$, and the corresponding FSS amplitudes, $\Lambda$ and $\Lambda_\nu$ (for both periodic and open BCs). All $\xi$ and $\Lambda$ that have appeared in the previous sections denoted values at the critical point.

A. Spin-orbit (symplectic) symmetry class

To compute the localization length at the LD transition in the symplectic class, we employed the so-called SU(2) model, a tight-binding model on the square lattice, with random on-site disorder and fully random SU(2) hopping.

1. Localization length (strip geometry)

We obtained the typical localization length from the smallest Lyapunov exponent of transfer matrices for very long Q1D lattices. We imposed hard-wall, i.e., reflecting BCs in the transverse direction and hence our Q1D samples had strip geometry. Our systems had a maximum size $M=128$ in the transverse direction. Figure 1(a) shows the FSS amplitude $\Lambda_\nu = 2\xi_0/M$ of the typical Q1D localization length as a function of the on-site disorder strength $W$ for various system sizes $M$ and at fixed energy $E=0$ (band center). The curves for the various system sizes intersect at different points re-
flecting large finite-size effects, in contrast to the case of periodic BCs.  

To determine the critical value of the FSS amplitude $\Lambda_{o,c}$, we performed a FSS analysis incorporating corrections to scaling arising from the leading irrelevant scaling variable.  

Specifically, we took a scaling function for the FSS amplitude of the form $\Lambda = F(\chi M^{1/\nu}, \xi M^{\nu})$, where $\chi$ is the relevant scaling variable, and $\xi$ is the leading irrelevant scaling variable whose scaling exponent $\gamma < 0$. The exponent $\nu$ characterizes the divergence of the 2D localization length $\xi$ upon approaching the LD transition point, $\xi \sim \chi^{-\nu}$. We expanded the scaling function around the critical point $W=W_c$, setting $\chi=(W-W_c)/W_c$,

$$\Lambda_o = \Lambda_{o,c} + \sum_{p=1}^{P} a_p(\chi M^{1/\nu})^p + M^{\nu} \sum_{q=0}^{Q} b_q(\chi M^{1/\nu})^q. \quad (41)$$

We fitted the numerical data to Eq. (41) with $P=Q=2$ by taking $W_c$, $a_p$, $b_q$, $\nu$, and $y$ as fitting parameters. We obtained

$$W_c = 6.192 \pm 0.007, \quad \Lambda_{o,c} = 1.50 \pm 0.01, \quad \nu = 2.79 \pm 0.03, \quad y = -1.03 \pm 0.03. \quad (42)$$

The details of the fitting are summarized in Table II. 

These results are in good agreement with those obtained by Asada et al.  

In our previous publication we reported the value $\alpha'_0 = 2.417 \pm 0.002$ for the surface exponent, which was obtained from numerical simulations on $L \times L$ lattices of system sizes up to $L=120$. We performed averaging over more than $6 \times 10^3$ disorder realizations. The lattices had periodic BC imposed in one of the two directions but open BC in the other direction, so our system had the geometry of a finite cylinder. Here we update the value for $\alpha'_0$ reported in our previous work.  

The surface exponent $\alpha'_0$ was obtained from the system size dependence of the wave-function amplitude in the vicinity of the boundary, according to

$$\langle \langle |\psi(x)|^2 \rangle \rangle \sim -\alpha'_0 \ln L + c. \quad (44)$$

Here $x = O(L^0)$, $L \gg 1$, and $c$ is a constant of order $L^0$. The double angular brackets represent both ensemble average and spatial average along the boundary of the cylinder in each disorder realization. First we tried a linear fitting to Eq. (44) of our numerical data for the left-hand side of Eq. (44), using system sizes $24 \leq L \leq 180$, with two fitting parameters $\alpha'_0$ and $c$. This resulted in the value

$$\alpha'_0 = 2.4195 \pm 0.0013. \quad (45)$$

Substitution of this value into Eq. (40) gave

$$\Lambda_{o,c} = 1.518 \pm 0.005. \quad (46)$$

This analysis, however, ignored corrections from irrelevant scaling variables and was not quite correct since we now know from the previous section that such corrections are appreciable for the FSS amplitude $\Lambda$ for open BC. We therefore reanalyzed the data, assuming scaling with corrections from the leading irrelevant variable.  

We define

$$A(x) := \frac{\langle \langle |\psi(x)|^2 \rangle \rangle}{\ln L} \sim -\alpha'_0 + \frac{1}{\ln L} (c + c'L), \quad (47)$$

where we take $y=-1$, as suggested by Eq. (42). The fitting of the same data to Eq. (47) yielded

$$\alpha'_0 = 2.429 \pm 0.006, \quad (48)$$

which leads to
with the help of Eq. (40). We see that the $\Lambda_{o,c}$ obtained from the transfer-matrix method (42) is consistent with these results. The value of $\alpha'_0$ reported in Eq. (48) has larger error bars, which needs to be improved in future numerical work.

B. Metal-to-$Z_2$-topological-insulator transition in quantum spin Hall systems

The $Z_2$ topological insulator is a time-reversal invariant topological insulator in two dimensions, which possesses a topologically protected Kramers pair of extended edge states at its boundaries. The $Z_2$ topological insulating states can be realized in materials with strong spin-orbit interactions, as evidenced by recent experiments on HgTe/(Hg,Cd)Te quantum wells. In the presence of disorder, this system undergoes a two-dimensional metal-insulator transition from a $Z_2$ topological insulator to a metal, as one changes the Fermi energy. On symmetry grounds, this LD transition is expected to belong to the spin-orbit (symplectic) symmetry class. Indeed, the critical exponent $\nu$ for the diverging localization length (a bulk property) $1/M$ at the metal-to-$Z_2$-topological-insulator transition is found to agree with the value obtained for the SU(2) model, which describes the metal-to-(ordinary) insulator transition in this symmetry class. Similar agreement is found for the multifractal exponents for critical wave functions in the bulk. However, the multifractal exponents characterizing wave-function amplitudes at the sample boundary turn out to be different at the two metal-insulator transitions.

Here we show that, at the metal-to-$Z_2$-topological-insulator transition, the FSS amplitude $\Lambda_{o,c}$ [Eq. (8)] for the typical Q1D correlation length in strip geometry, is related by conformal invariance to the boundary multifractal exponent $\alpha'_0$ at the same transition.

1. Localization length (strip geometry)

To compute the localization length at the metal-to-$Z_2$-topological-insulator transition, we employed the quantum spin Hall network model. An important parameter in this network model is the one controlling the probability of tunneling at the nodes of the network, which we denote by $X$. The numerical results shown below were obtained at the critical point $X_c=0.971$ with fully random SU(2) spin-rotation symmetry on each link. Figure 2 shows the dependence of the FSS amplitude $\Lambda_x(M) = 2\xi_x(M)/M$ of the typical Q1D localization length $\xi_x(M)$ on a strip of width $M$ ($M=8, 10, 12, 16, 24, 32, 48, 64$). Here $M$ is the number of nodes of the network model in the transverse direction across the Q1D strip. This corresponds to transfer matrices of size $4M \times 4M$. In order to find the critical value $\Lambda_{o,c}$ of the FSS amplitude $\Lambda_x$ in the large $M$ limit, we assumed that $\Lambda_x$ at $X=X_c$ has a power-law finite-size correction due to a leading irrelevant variable with dimension $y<0$

$$\Lambda_x(X=X_c) = \Lambda_{o,c} + b_0 M^y.$$  

(50)

Fitting the data to this form (see Fig. 2), we obtained

$$\Lambda_{o,c} = 7.20 \pm 0.01$$

(51)

with $y=-0.81 \pm 0.08$ and $b_0=-1.0 \pm 0.1$. The details of the fitting are summarized in Table II.

2. Surface multifractal exponent $\alpha'_0$

The surface multifractal exponent at the metal-to-$Z_2$-topological-insulator transition was obtained in Ref. 38. By using larger system sizes this value was recently improved in Ref. 39 to

$$\alpha'_0 = 2.091 \pm 0.002. \quad \text{(52)}$$

Substituting the improved value into Eq. (40) yields the FSS amplitude

$$\Lambda_{o,c} = 7.00 \pm 0.15. \quad \text{(53)}$$

This value is consistent with Eq. (51). The larger error bar in Eq. (53) results from the fact that the denominator in Eq. (40) (with $\chi_p=0$) contains $\alpha'_0=2=0.091 \pm 0.002$. Neither of the numerical analyses in Refs. 38 and 39, used to obtain Eq. (52), included effects of the leading irrelevant variable, in contrast to Eq. (51). These effects may influence the value of $\alpha'_0$ and possibly result in better agreement with Eq. (51).

C. Plateau transition in the integer quantum-Hall effect

To compute the localization length $\xi_x$ and the surface multifractal exponent $\alpha'_0$ at the plateau transition in the IQH effect, we employed the Chalker-Coddington network model in strip geometry with $M$ nodes in the transverse direction across the strip. This corresponds to transfer matrices of size $2M \times 2M$. The plateau transition is reached by tuning a parameter $\theta$ which controls the tunneling probability at the nodes of the network model. For this model the critical value $\theta_c$ is known exactly.

1. Localization length (strip geometry)

The typical localization length $\xi_x$ in Q1D strip geometry was computed numerically from the smallest Lyapunov ex-
To find the critical value of the FSS amplitude increases, indicating the presence of finite-size corrections. As seen from Fig. 3, the critical exponent of the diverging function of the network model tunneling parameter that we studied, was \( M = 64 \). Figure 3(a) shows the FSS amplitude \( \Lambda_{\alpha} = 2 \xi_{c} / M \) of the typical localization length as a function of the network model tunneling parameter \( \theta \) for various transverse system sizes \( M \).\(^{40} \) For \( \theta > \theta_{c} \), the network model is in the quantum-Hall phase.\(^{42} \) As seen from Fig. 3, the crossing point of the curves moves towards \( \theta = \theta_{c} \) as \( M \) increases, indicating the presence of finite-size corrections. To find the critical value of the FSS amplitude \( \Lambda_{\alpha} \) of the typical 1D correlation length in the large \( M \) limit, we fitted the data to Eq. (50) [see the inset of Fig. 3(a)], to obtain

\[
\Lambda_{\alpha,c} = 1.624 \pm 0.002, \quad y = -1.29 \pm 0.04, \quad b_{0} = 1.26 \pm 0.7, \quad b_{1} = 2.016 \pm 0.086, \quad b_{2} = -0.73 \pm 0.26.
\]

FIG. 3. (Color online) (a) Dependence of \( \Lambda_{\alpha} \) on the node parameter \( \theta \) in the Chalker-Coddington model of the strip geometry. The vertical dashed line indicates the critical point \( \theta = \theta_{c} \). Inset: \( M \) dependence of \( \Lambda_{\alpha} \) at \( \theta = \theta_{c} \); the solid curve is a fit to Eq. (50). (b) Scaling plot from FSS analysis including corrections from the leading irrelevant scaling variable. The parameters used for the plot are \( \nu = 2.55 \pm 0.01, \quad a_{1} = 2.518 \pm 0.016, \quad a_{2} = 2.179 \pm 0.027, \quad a_{o} = 1.393 \pm 0.051, \quad b_{0} = 1.26 \pm 0.7, \quad b_{1} = 2.016 \pm 0.086, \quad \) and \( b_{2} = -0.73 \pm 0.26 \).

ponent of the transfer matrices. The largest system size (the number of network model nodes in the transverse direction) that we studied, was \( M = 64 \). Figure 3(a) shows the FSS amplitude \( \Lambda_{\alpha} = 2 \xi_{c} / M \) of the typical localization length as a function of the network model tunneling parameter \( \theta \) for various transverse system sizes \( M \).\(^{40} \) For \( \theta > \theta_{c} \), the network model is in the quantum-Hall phase.\(^{42} \) As seen from Fig. 3, the crossing point of the curves moves towards \( \theta = \theta_{c} \) as \( M \) increases, indicating the presence of finite-size corrections. To find the critical value of the FSS amplitude \( \Lambda_{\alpha} \) of the typical 1D correlation length in the large \( M \) limit, we fitted the data to Eq. (50) [see the inset of Fig. 3(a)], to obtain

\[
\Lambda_{\alpha,c} = 1.624 \pm 0.002, \quad y = -1.29 \pm 0.04, \quad b_{0} = 1.26 \pm 0.7. \quad \text{The details of the fitting are summarized in Table II.} \quad \text{Figure 3(b) shows the data collapse from the FSS analysis using Eqs. (41) and (43) with} \quad \chi = (\theta - \theta_{c}) / \theta_{c} \quad \text{and the values of} \quad \Lambda_{\alpha,c}, \quad y, \quad \text{and} \quad b_{0}. \quad \text{This FSS analysis also yielded} \quad \nu = 2.55 \pm 0.01 \quad \text{for} \quad \text{the critical exponent of the diverging (2D bulk)} \quad \text{localization length, which is close to the value obtained in a recent large-scale numerical study,} \quad \nu = 2.593 \pm 0.006.\(^{43} \)

2. Surface multifractal exponent \( \alpha_{s}^{0} \)

The surface multifractal exponent \( \alpha_{s}^{0} \) at the plateau transition was recently obtained by the present authors\(^{44} \) and by Evers et al.\(^{45} \) It was found in these works that the multifractal analysis for the Chalker-Coddington model suffers from large finite-size corrections. To reduce these corrections, we have used, in the multifractal scaling analysis in Ref. 44, numerical data obtained only for large system sizes. Here we used an alternative approach by taking into account corrections to scaling arising from a leading irrelevant scaling variable using Eq. (47).

The geometry of the Chalker-Coddington network model that we used is shown in Fig. 4. There are two types of nodes forming two sublattices (denoted A and B in the figure), such that the A sublattice has the size \( L \times 2L \) (i.e., \( L = 3 \) in the figure). The links in the network form zigzag-shaped rows and columns; there are 2L such rows and 2L such columns so that the total number of links is \( 4L^{2} \). Integer \( x \) and \( y \) coordinates are assigned to the centers of links. We imposed periodic BC in the vertical \( y \) direction and reflecting BC in the horizontal \( x \) direction. The links in the first and the last columns at \( x = 1 \) and \( x = 2L \) are called the edge links. The discrete time evolution of wave functions defined on links of the network model is governed by a unitary evolution operator \( U \) for one discrete time step, which is determined by the scattering \( S \) matrices at the nodes of the network model.\(^{46} \) In our case this operator is a \( 4L^{2} \times 4L^{2} \) unitary matrix. For each disorder realization, we obtained one critical wave function that is the eigenvector of \( U \) at \( \theta = \theta_{c} \) and whose eigenvalue is closest to unity among all the eigenvectors. The largest system size we studied was \( L = 180 \), and the disorder average was taken over \( 3 \times 10^{5} \) realizations for \( L \leq 60 \), over \( 5 \times 10^{5} \) realizations for \( L = 80 \), and over \( 2 \times 10^{5} \) realizations for \( L = 120, 180 \).

Figure 5(a) shows the \( x \) dependence of \( \langle |\ln| \psi(x) |^{2} \rangle \rangle \), where the double angular brackets stand for both the average over disorder realizations and the spatial average along the


where the error bars reflect only statistical errors. This result is consistent with that of Ref. 44 ($a_0'=2.386\pm0.004$). Figure 5(b) shows that fitting of $A(x=1)$ and $A(x=2)$ gives similar values of $a_0'$. Substituting Eq. (55) into Eq. (40) yields

$$a_0'=2.385\pm0.003,$$

which should be compared with $a_0'=1.624\pm0.002$ [Eq. (54)] obtained from the transfer-matrix calculation. As we see in Fig. 5(b), finite-size corrections to $A$ and $\Lambda_p$ are still quite large at $L=180$. This makes the extrapolation of these quantities to $L\to\infty$ difficult; we cannot exclude the possibility of having systematic errors in addition to the statistical errors included in Eqs. (54) and (56). Given the presence of this uncertainty, we conclude that our numerical results are consistent with Eqs. (7) and (40).

D. Spin quantum-Hall plateau transition of BdG quasiparticles in symmetry class C

In this section we discuss the verification of Eqs. (6) and (7) for symmetry class C, which is known to possess a vanishing critical DOS ($x_p>0$). In our simulations we used an appropriate generalization of the Chalker-Coddington network model,\(^7\) which we refer to as the class C network model. This model has a control parameter $\epsilon$ (in the notation of Ref. 47), and is critical at $\epsilon=0$. Exact values for critical exponents, $\nu=4/3$ and $x_p=1/4$, were obtained through mapping to classical percolation.\(^{21,22}\) The exact values of the bulk\(^{23}\) and surface\(^{24}\) multifractal wave-function exponents $x_{q,0}$ are also known at $q=2,3$. However, exact results for the FSS amplitudes of the typical Q1D correlation lengths, $\Lambda_{p,c}$ and $\Lambda_{a,c}$, and the typical wave function scaling exponents $\omega^{p,c}_{0}$ are not available.

I. Localization length (cylinder and strip geometries)

We numerically obtained the FSS amplitudes of the typical Q1D localization length of the class C network model for both cylinder and strip geometries. A previous numerical study\(^{47}\) of FSS of the typical localization length in cylinder geometry did not report the value of $\Lambda_{p,c}$. Here we present results for the FSS amplitudes $\Lambda_{p,c}$ and $\Lambda_{a,c}$ corresponding to cylinder and strip geometries, respectively.

Cylinder geometry. Figure 6(a) shows the dependence of the FSS amplitude $\Lambda_p$ of the typical Q1D correlation length on the parameter $\epsilon$ for various values of the transverse width $M$, obtained in cylinder geometry. The FSS amplitude $\Lambda_p$ is symmetric about the critical point $\epsilon_c=0$ when periodic BCs are imposed. Hence in the FSS analysis we have to use an expansion in even powers of $\epsilon$,

$$\Lambda_p=\Lambda_{p,+}\sum_{p=1}^{P}a_{2p}(\epsilon M^{1/\nu})^{2p}+\Lambda_{p,-}\sum_{q=0}^{Q}b_{2q}(\epsilon M^{1/\nu})^{2q}.\quad(57)$$

The result of fitting of the data in Fig. 6(a) to Eq. (57) is shown in Fig. 6(b). The dependence of $\Lambda_p(\epsilon_c)$ on the width $M$ at the critical point $\epsilon_c=0$ is plotted in Fig. 6(c). We obtained

$$\Lambda_{p,c}=0.8189\pm0.0004\quad(58)$$

and $\nu=1.335\pm0.016$. The details of the fitting are summarized in Table II. The latter result is consistent with the exact value $\nu=4/3$, indicating good accuracy of our numerical results.
A critical point is known to be located at \( \epsilon_c = 0 \). For several values of \( M \) in the class C network model of strip geometry, the critical point is located at \( \epsilon_c = 0 \). The dependence of \( \tilde{\Lambda}_p \) on \( \epsilon \) for several values of \( M \) with \( \epsilon = 0 \) is shown in Fig. 7(a). The critical point is located at \( \epsilon_c = 0 \). The dependence of \( \Lambda_o \) on \( \epsilon \) for several values of \( M \) with \( \epsilon = 0 \) is shown in Fig. 7(b) and 7(c). These values are consistent with the values presented in Eqs. (58) and (59) obtained by our FSS analysis.

**IV. CONCLUSIONS**

In this paper we have generalized the formula relating the multifractal exponent \( \alpha_0 \) of the typical wave-function amplitude in a 2D sample to the FSS amplitude \( \Lambda_o \) of the typical localization length in a 1D sample. Our generalization is twofold, resulting in Eqs. (6) and (7). Our Eq. (6) extends the relation to unconventional symmetry classes where the global density of states vanishes at criticality. Our Eq. (7) extends the relation to the case when the 1D sample has strip geometry, instead of cylinder geometry which was always considered in earlier studies.

Substitution of these values into Eqs. (29) and (40), respectively, with \( x_p = 1/4 \) yields

\[
\Lambda_o = 0.8225, \quad \alpha_0 = 1.105.
\]  

These values are consistent with the values presented in Eqs. (58) and (59) obtained by our FSS analysis.
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integer quantum-Hall plateau transition, and (d) the spin quantum-Hall plateau transition. Our numerical results are summarized in Tables I and II.

ACKNOWLEDGMENTS

We acknowledge helpful discussions with A. Mirlin, C. Mudry, and S. Ryu. This work was partly supported by the Next Generation Super Computing Project, Nanoscience Program from MEXT, Japan. Numerical calculations were performed on the RIKEN Super Combined Cluster System. H.O. is supported by JSPS. The work of A.F. was supported by a Grant-in-Aid for Scientific Research from MEXT and JSPS, Japan (No. 16GS0219 and No. 21540332). I.A.G. was partially supported by NSF under Grant No. DMR-0448820 and NSF MRSEC Grant No. DMR-0213745. The work of A.W.W.L. was supported in part by NSF under Grant No. DMR-0706140.

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15 At any conventional nonrandom second-order phase transition in two dimensions, all scalar scaling operators which possess scaling dimensions smaller than two are examples of conformal primary operators.
26 Not to be confused with the “spin quantum-Hall” transition in symmetry class C, mentioned above.
27 These assumptions have been numerically verified in Ref. 13 for one of the standard symmetry classes which possess a LD transition in 2D, namely, for the 2D metal-insulator transition in the spin-orbit (symplectic) class.
28 Note that this boundary condition has nothing to do with the open (reflecting) boundary condition we consider in Sec. II B.
29 In a supersymmetric sigma model arbitrary real moments of the local density of states can be obtained by exploiting the noncompact sector of the theory.
31 It is known (Ref. 5) that the exponents $\nu_q$ are positive only within some range $0 < q < q_u$. Outside this range $\nu_q < 0$, which leads to LDOS moments $[\rho(w)]^q$ that are exponentially growing away from the cylinder boundary. This can be explained by the following argument. In cylinder geometry, the wave functions are localized, and the level broadening $\eta$ created by the metallic lead attached at the edge $w=L$ is strongly coordinate dependent. Indeed, the overlap of a wave-function localized at a point $w$ away from the cylinder edge with the extended wave functions in the lead is exponentially small, which results in an escape rate, that is, a level broadening that is exponentially decaying away from the edge: $\eta(u) \propto e^{-|L-u|/\xi}$. Here $\xi$ is the typical localization length. Thus, away from the edge, the profile of the LDOS (as a function of energy) consists of well-resolved narrow Lorentzian peaks [see Eq. (10)] of width of order $\eta(u)$ and height proportional to $1/\eta(u)$. Only within the range $0 < q < q_u$ corresponding to $\nu_q > 0$ are the moments of the LDOS determined by the minima of the LDOS profile [where the LDOS is proportional to $\eta(u)$]. For other values of $q$ the dominant contribution to the moments of LDOS comes from the maxima in the LDOS profile. In the described situation the behavior of the moments of the wave functions is very different from that of the moments of the LDOS. However, we can directly relate the latter to the transmission through the cylinder from the metallic lead. Essentially, the transmission will be exponentially small away from the edge of the cylinder if the energy of the incoming wave (in the lead) corresponds to a minimum in the LDOS profile but will be of order one (it cannot be bigger) if the energy is in resonance with one of the broadened energy levels (that is, a peak in the LDOS profile).
32 The dependence on $z$, which encodes the distance from the boundary, is suppressed here.
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35 C. L. Kane and E. J. Mele, Phys. Rev. Lett. 95, 146802 (2005); 95, 226801 (2005).
42 In this phase $\Lambda_o$ increases with $M$, due to the presence of edge states, which increase the conductance (similar to the case of a metallic phase).
43 K. Slevin and T. Ohtsuki, Phys. Rev. B 80, 041304(R) (2009).